Algebra-IV, IInd Year BMath
Mid-Semestral Exam
March, 2022

Time: 2 hours

Notations: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{F}_{p^n}$ for a prime p denotes the finite field with p^n elements.

Answer all questions. Provide complete justification for all your answers.

(1) (i) Let K be a field extension of F and $a \in K$ be such that $[F(a) : F]$ is odd. Show that $F(a) = F(a^2)$.
(ii) Give an example to show that this statement may be false if the degree of $F(a)$ over F is even. (2+4)

(2) (i) Prove that if K_1 and K_2 are two finite extensions of a field F contained in some field K, then
$$[K_1K_2 : F] \leq [K_1 : F][K_2 : F],$$
where K_1K_2 is the composite field of K_1 and K_2.
(ii) Also show that equality holds if $[K_1 : F]$ and $[K_2 : F]$ are relatively prime. (4+2)

(3) Let F be a finite field of characteristic p and let $f(x) \in F[x]$ be an irreducible polynomial.
(i) Prove that there exists a $n \in \mathbb{N}$ such that $f(x)$ divides the polynomial $x^{p^n} - x$ in $F[x]$.
(ii) Show that $f(x)$ is separable. (3+3)

(4) Let $\Phi_n(x)$ denote the n-th cyclotomic polynomial. Then prove that
(i) For any $n \in \mathbb{N}$, $x^n - 1 = \Pi_{d/n}\Phi_d(x)$
(ii) For n odd, $n > 1$,
$$\Phi_{2n}(x) = \Phi_n(-x).$$
(2+4)

(5) (i) Define Galois extensions.
(ii) Show that for a prime p and $n \in \mathbb{N}$, $\mathbb{F}_{p^n}$ over $\mathbb{F}_p$ is a Galois extension.
(iii) Show that the Galois group $\mathrm{Gal}(\mathbb{F}_{p^n}|\mathbb{F}_p)$ is cyclic of order n. Exhibit a generator of the Galois group. (1+2+3)